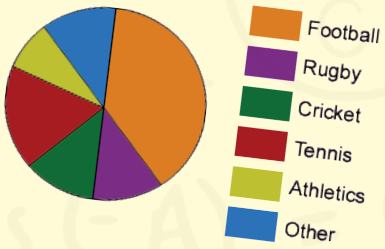
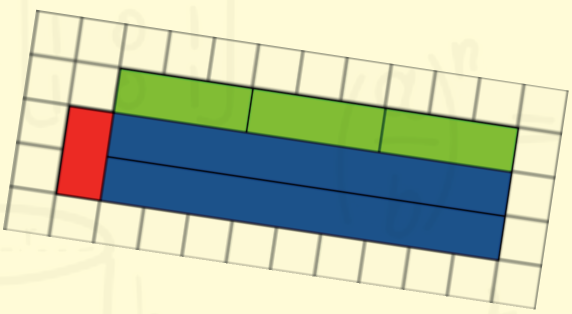
Peter Mattock

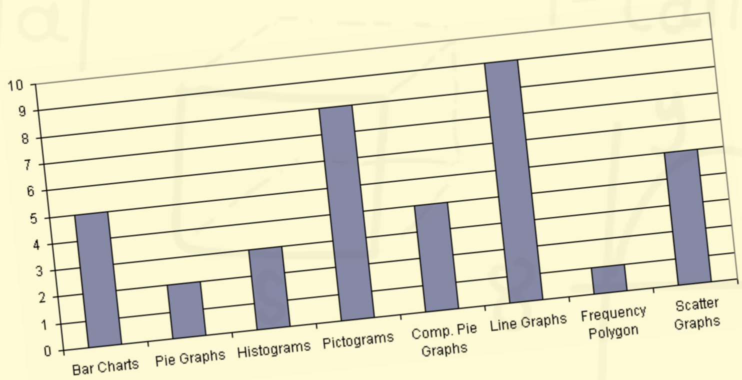


$$5y^{-4.7} + 2y^{-4.7} - y^{-4.7}$$

Conceptual Maths

Teaching 'about' (rather than just 'how to do') mathematics in schools





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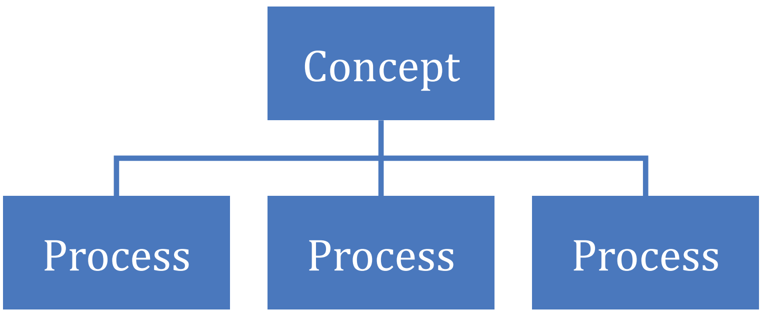
Introduction

What is mathematics? Interestingly, although mathematics has been an integral part of the school curriculum for the best part of the last 75 years or so,[1] there is little consensus on the answer to this question amongst teachers of mathematics. Some people will say that mathematics is a body of connected knowledge, others that it is a way of behaving and making sense of the world, whilst others will say it is a collection of theorems based on fundamental axioms. Some may see this lack of consensus as problematic – we can hardly ensure that learners of mathematics are getting a consistent experience if their teachers don't all have the same idea of what they are teaching! However, for me, a consensus amongst maths educators about what mathematics is isn't as important as what mathematics is not. One thing (in my opinion) that mathematics definitely is not, is a collection of procedures.

That is not to say that procedures aren't important in mathematics; simply that if all one learns about mathematics is how to complete procedures, then one hasn't really learnt a lot about mathematics. Primarily this is because there are (nearly always) many different procedures that will accomplish the same result, with the choice of procedure largely dependent on a mixture of efficiency and what the teacher is familiar with or prefers. Jo Morgan's excellent *A Compendium of Mathematical Methods* highlights some of the multitude of procedures that exist for doing things like multiplying large numbers or solving equations.[2] But a pupil could learn every method in the book and still not have learnt much about mathematics. To learn about mathematics, one has to go deeper, beyond the procedures, and into the structure of its different concepts. In *Mathematics Counts*, the Committee of Inquiry into the Teaching of Mathematics in Schools (the Cockcroft Report), states: '*Conceptual structures* are richly interconnected bodies of knowledge, including the routines required for the exercise of skills. It is these which make up the substance of mathematical knowledge stored in the long term memory.'[3]

Concepts are at the heart of the study of mathematics. They are the ideas that remain constant whenever they are encountered but that combine and build upon each other to create the mathematical universe. The structure of each concept is what gives rise to the procedures and processes that are used in calculation and problem solving. In learning about the structure of each concept, a learner of mathematics can make sense of how different processes are doing what they do, using them flexibly as need demands. A simple image to capture this relationship might look like this:



1 M. McCourt, A Brief History of Mathematics Education in England, *Emaths* [blog] (29 December 2017). Available at: https://www.emaths.co.uk/index.php/blog/item/a-brief-history-of-mathematics-education-in-england.

2 J. Morgan, *A Compendium of Mathematical Methods* (Woodbridge: John Catt Educational Ltd, 2019).

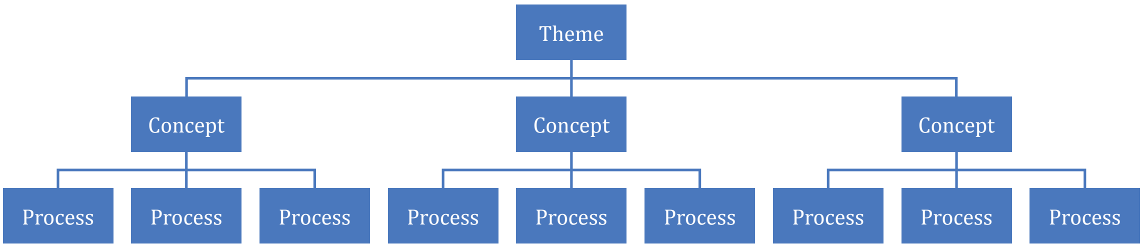
3 W. Cockcroft, *Mathematics Counts: Report of the Committee of Inquiry into the Teaching of Mathematics in Schools under the Chairmanship of Dr W. H. Cockcroft* [Cockcroft Report] (HMSO, 1982), p. 71. Available at: http://www.educationengland.org.uk/documents/cockcroft/cockcroft1982.html.

However, this model ignores two important aspects of mathematics: the interplay between concepts and overarching themes.

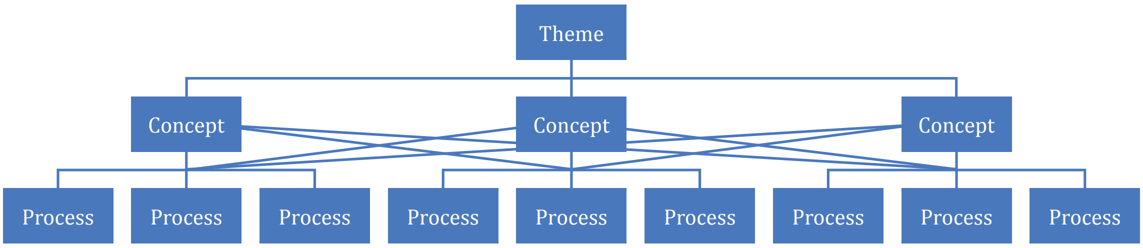
In their fantastic book *Developing Thinking in Algebra*, John Mason, Alan Graham and Sue Johnston-Wilder put forward five mathematical themes:

- Freedom and constraint.
- Doing and undoing.
- Extending and restricting.
- Invariance and change.
- Multiple interpretations.[4]

These themes appear across different mathematical concepts (the last will appear a lot throughout this book) and provide key touchpoints that learners (and teachers) can keep coming back to when they study different concepts. That complicates the model slightly:



But the flowchart above still doesn't capture the interplay between concepts or how they can come together to build the mathematical universe. One might more accurately adapt the above model to look something like this:



But even this is too simple; some concepts derive from others, some processes bring multiple concepts into play and several themes appear in each concept. The real model is likely to be three-dimensional or higher in order to capture all of the links between all of the themes, concepts and processes. This, of course, explains why it is so difficult to design a curriculum for mathematics even though it is essentially a hierarchical subject – concepts are introduced, then disappear for a while before reappearing in conjunction with other concepts that have been developed in the meantime. It also means that capturing this in text form is incredibly complex, with lots of back and forth between different concepts. In order to support this, each concept will include details such as:

1 Concept – what the concept being explored is.

4 J. Mason, A. Graham and S. Johnston-Wilder, *Developing Thinking in Algebra* (London: SAGE Publications, 2005).

2 Prerequisites for each concept – other concepts or parts of a concept that this concept requires to be secure before the given concept is introduced.

3 Linked concepts – other concepts that will come into play with the given concept when developing certain procedures or other aspects of the concept.

4 Good interpretations – good ways of thinking about/representing the given concept.

5 Good questions to ask – questions that can be asked or tasks that can be given to learners to support understanding of the structure of the concept.

6 Procedures – procedures that mainly arise from, or are associated with, the given concept and how to make their link to the concept explicit.

The first three of the above will typically be listed at the start of each section (as well as highlighted when they appear) and the rest will be explored in the main body of the section. It won't be necessary to look at all of these details for all concepts, and for some there may be other details that we will look at, but this will be the essence of what will be addressed for each concept.

In order to try and make the exploration of the mathematical concepts more manageable, we will group them into broader topic areas. These are not necessarily the sort of topic areas we might use with children (although some might be), and as explained there will be plenty of crossover from concepts in one area to concepts in another (there would be no matter how you defined the topic area); however, it will allow for a grouping of broadly similar and strongly related concepts.

We will start with an examination of the concept of number, which will include its generalisation into algebra. We will then move on to looking at the standard numerical operations in three parts: addition/subtraction, multiplication and multiples, and then division and factors. We will follow this with an examination of ideas around equivalence and equality before shifting our attention to proportionality and then functionality. From here we will begin to look at concepts in the realm of geometry, including measures, accuracy, shape and transformation. We will finish in the realms of chance, data and graphing/charting.

In my first book, *Visible Maths*, I concentrated on how the use of representations and manipulatives can provide a window into some of these mathematical structures and can support pupils in creating some of these connections by being able to draw on particular images and tools that could represent mathematical concepts whenever a pupil was working with them. Whilst there will be some crossover in this book, and readers of *Visible Maths* will find some things familiar, my aim with this book is to go broader, but not necessarily as deep, in all areas. *Visible Maths* (I hope) really got into the detail of some very specific concepts, at least in terms of the inherent mathematical structures of those ideas and how they can be manipulated. In this book, I will provide more of an overview of more concepts and include teaching ideas that are not necessarily related to the structure (such as good questions or activities that highlight aspects of the concept). My hope is that people will see *Visible Maths* as a companion to this work, so that in this book they find reference to all the concepts they will be teaching across primary and secondary schools, along with key advice/suggestions for how to ensure they teach this idea in a way that makes its structure explicit so it can be linked with other ideas. In a complementary fashion, in *Visible Maths* they can delve in depth into certain important concepts and look in detail at how good representations and manipulatives can be used to really get into a concept with pupils.

In recent years there has been, in some circles of mathematics education, a strong move back to trying to secure 'procedural fluency' prior to developing 'conceptual understanding'.

Many influential maths teachers are suggesting that learners can gain greater insight into the structure of a concept if they have first reached the point where they are very comfortable with the procedures associated with the concept. However, I am sceptical of this for two reasons:

1 Much of my experience of maths education to date has been of a very instrumental approach in which pupils are often practising procedures for much of the time in the classroom that they don't spend listening to their teacher.[5] Whilst I can see ways to improve this practice so that learners take more from the experience, I don't see it providing the gains in pupil understanding or in their motivation to continue studying mathematics education.

2 As I have intimated, the procedures attached to different concepts actually arise from the structure of the concept itself. To rely on knowledge of the procedure to provide understanding of the concepts seems to be backwards in approach. In addition, because there are many different procedures associated with each concept, it would seem to be time-intensive to have to study many of them to the point of automaticity before being able to use this experience to gain a window into the underpinning structure. Instead, first securing the structure and then exploiting the structure to gain insight into the associated procedures would seem much more logical.

This move back towards procedural fluency is generally attributed to a reaction against the perceived dominance of constructivist approaches to education[6] (which are thought to be linked with a more discovery-based approach to learning) in the late 1990s and first decade of the 21st century. Some feel that this has led to learners being held back as they haven't had their learning directed adequately by a teacher. For me, though, if this is the case, the cure is not to move to teaching procedures. Teaching structure is compatible with both constructivist and didactic approaches to education because it concerns itself with the content to be taught rather than how to go about teaching it. That is what I aim to show in this book: how exposing learners to mathematical structure can ensure they achieve both procedural fluency and conceptual understanding, whether your preferred pedagogy is to teach it explicitly or to offer learners activities to discover this structure through inquiry. Hopefully, in reading this book, teachers will become familiar with the underlying structure for the key concepts in school-level mathematics and will then be able to use this knowledge to support learners in making sense of the content they study. Whether you support learners in constructing that sense for themselves or explicitly teach good ways of making sense of concepts, ensuring learners can make sense of mathematics concepts puts them in a much better place to see the connections between the things they study.

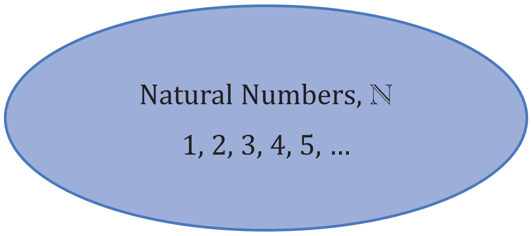
The point about making connections is important. In the last few years, cognitive science has had increasing exposure to teachers and is influencing practice on a larger scale. One of the key ideas in cognitive science is that of a schema. A schema represents our knowledge in a particular area, and how it is connected. If we wish to learn something new in that area, we have to be able to connect it to our existing schema. By teaching about the structure of concepts, these connections become much easier to highlight because the concept is recognisable every time it reappears.[7] What I would hope is that, having read this book, teachers feel able to support learners in recognising the structure behind different mathematical concepts and help them assimilate or accommodate new knowledge of the concept into their schema.

5 R. R. Skemp, Relational Understanding and Instrumental Understanding, *Mathematics Teaching*, 77 (1976), 20–26. Available at: https://www.lancsngfl.ac.uk/secondary/math/download/file/PDF/Skemp%20Full%20Article.pdf.
6 Learning Theories, Constructivism (n.d.). Available at: https://www.learning-theories.com/constructivism.html.
7 Learning Theories, Schema Theory (n.d.). Available at: https://www.learning-theories.org/doku.php?id=learning_theories:schema_theory.

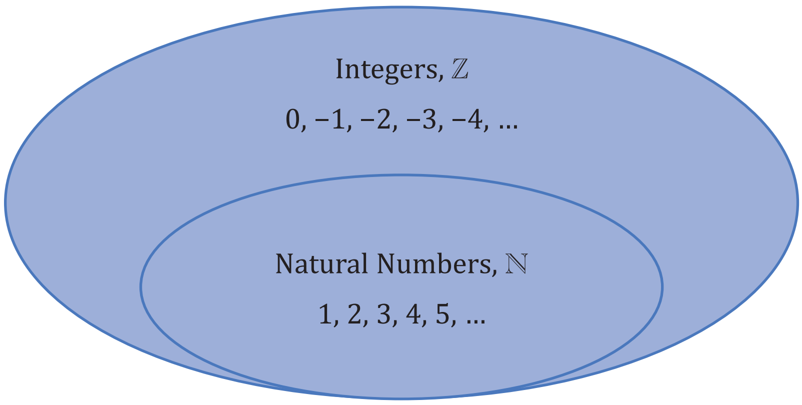
Chapter 1

Number

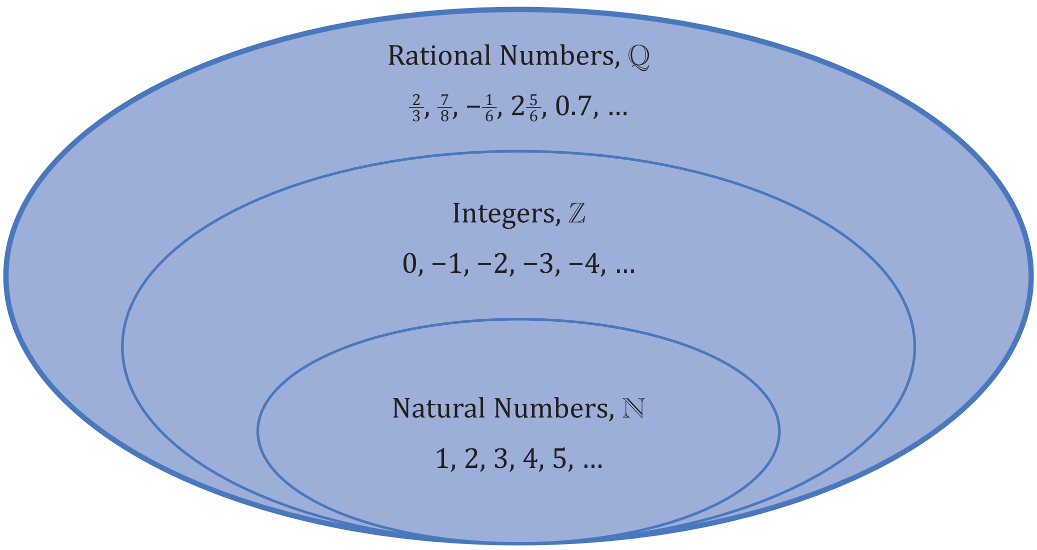
There are many different 'types' of numbers. A common way to view these is as sets, with each set becoming a subset of a further set. Mostly this starts with the natural numbers, which are the whole numbers greater than 0 (also called the positive integers).



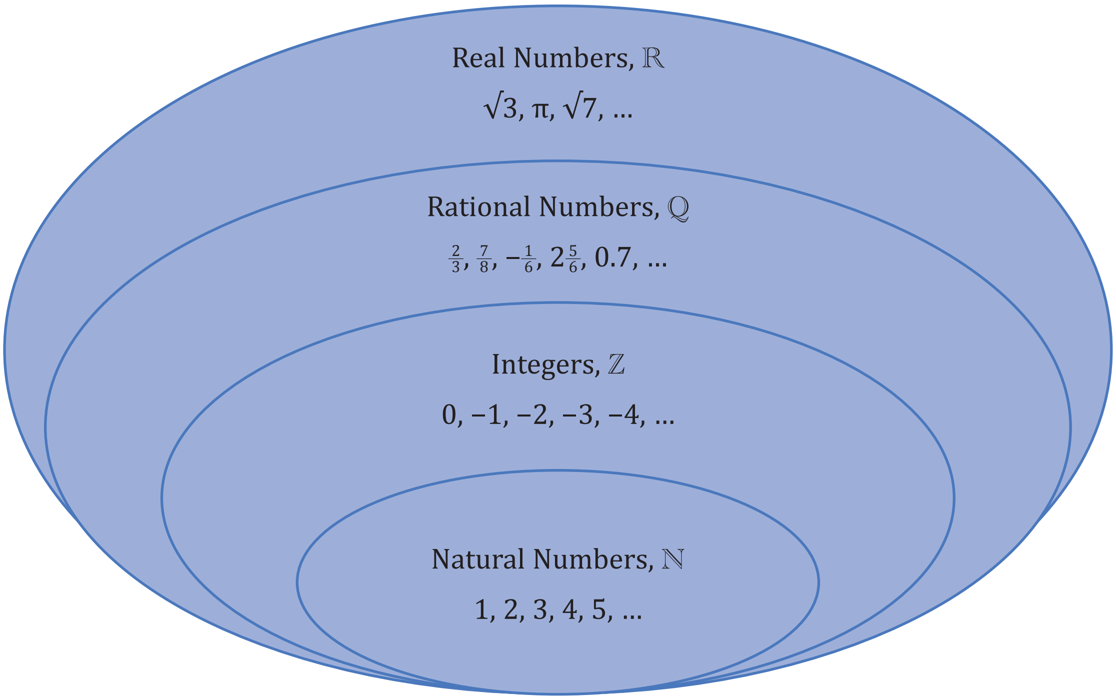
We then extend these to include 0 and the negative integers, creating the set of all integers.



After this we include fractions and decimals, creating a set collectively known as the quotients or rational numbers.



The next layer includes the irrational numbers, which creates the set of all real numbers.



This is the limit of school-level number in the UK, although there is a level beyond this that is explored post-16.

Each of these types of numbers have different links and prerequisite concepts; however, what they all have in common is that there are two ways to make sense of them – as **discrete** objects or as **continuous** measures.[1] These go to the very heart of what makes a number. In the early 20th century, there was a movement in philosophy that sought to place mathematics on firm logical footings. A major contributor to this movement was the eminent philosopher Bertrand Russell. In his work *Principia Mathematica* with Alfred North Whitehead, he defined what it means to be a number in terms of all sets of objects that exhibit the same property as that number.[2] So, the number 'four' is defined as all the sets that have four objects, such as the number of prime numbers less than 10 or the number of legs a cat is usually born with, and all of these are associated with the numeral '4'. This was an attempt to put the concept of 'number' on a firm logical footing. Whilst probably overkill for most school-level pupils (although a very interesting question to pose to pupils is 'What is 3?'), it nonetheless speaks to how people *see* numbers. Making sense of numbers is the precursor to pretty much all mathematical learning, and so it follows that exploring these different types of numbers and useful ways of thinking about them is a good place to start.

1 You will notice whilst reading the book that some key mathematical terms are presented in **bold** – for your convenience these terms are defined in a glossary at the back of the book.
2 See https://www.britannica.com/topic/history-of-logic/Logic-since-1900#ref535751.

Concept: natural number

Prerequisites: None (although some research suggests that informal ideas around spatial reasoning can drastically improve development of early number sense[3]*).*

Linked concepts: Addition and multiplication are linked directly to the development of the natural numbers, and division will also be important for the development of place value. Virtually all other concepts have natural numbers as a prerequisite.

Children often learn to count prior to starting formal education. Learning to repeat the sequence 'one, two, three' and so on up to 'ten', however, does not imply that a learner has an appreciation of number. What this shows is an appreciation of **order**. The learner that knows that 'three' follows 'two' does not necessarily know that 'three' represents a larger quantity than 'two' or that 'three' is 'one more' than 'two'. What is essential in learning about number is to relate the words 'one', 'two', 'three' with the numerals '1', '2', '3' – and to the actual values that these words/numerals represent.

This process typically starts in early years, with the use of concrete objects. However, as suggested in *Key Understandings in Mathematics Learning. Paper 2: Understanding Whole Numbers*, the relating of number and quantity can take 'three to four years'.[4] This highlights the importance of continuing the use of concrete representations throughout Key Stage 1 and the importance of using them to continue to develop and reinforce learners' knowledge or the links between numerals and quantities. These concrete objects will generally start out as truly representative of the context a learner is working on. For example, if talking about 'How many toys?', the concrete objects will be actual toys. From here, learners will need to make two transitions:

1. The partial abstraction of using standard concrete objects to represent quantities in a one-to-one relationship.
2. The further abstraction of using standard concrete objects to represent quantities in a one-to-many relationship.

Of course, each learner will be ready for these transitions in their own time, and care must be taken not to rush learners through these transitions.

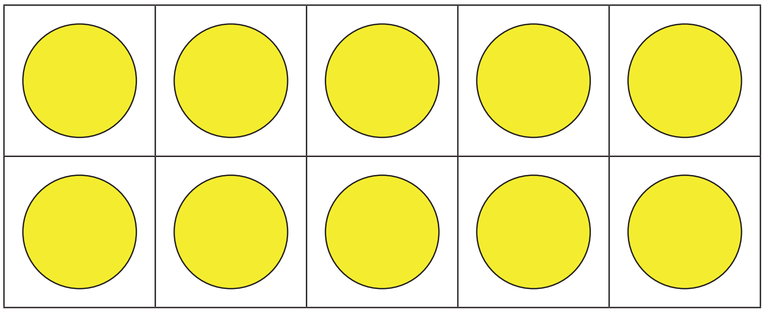
There are many different choices for the 'standard' concrete object to represent quantities in a one-to-one relationship. These can be counters, counting sticks, beads on a rekenrek, cubes; the list is substantial. This object (or objects, as it is a good idea to introduce more than one at different points) becomes the standard to represent a quantity in every situation. If the learner is working on a problem involving toys, one object becomes one toy. If they are working on a problem involving sweets, then one object is one sweet. In this way, the objects *become* the quantity; they represent the quantity in all situations. This is the first step to working with numbers in the abstract, to manipulate them, compare them and operate with them.

The obvious difficulty then arises when pupils start to work with larger and larger values. It is fine to represent 3 as three counters, but to represent 33 in this way is woefully inefficient. Worse, it doesn't actually help in making sense of 33 as a concept – there are simply

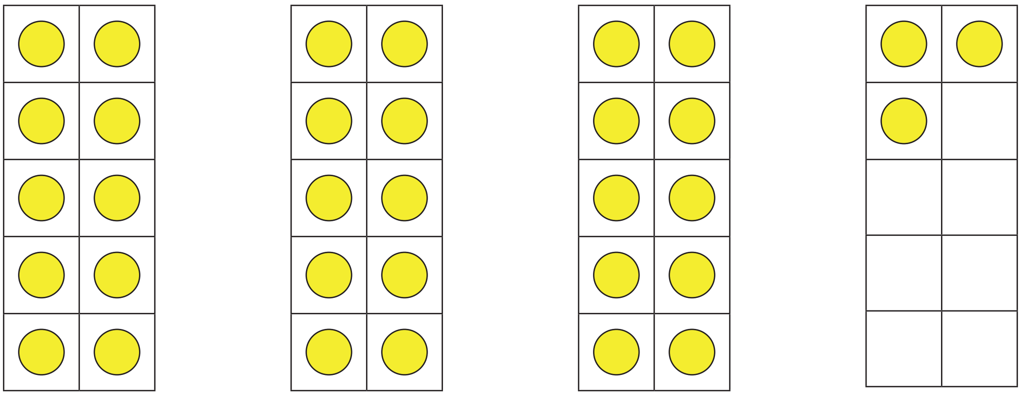
3 H. J. Williams, Mathematics in the Early Years: What Matters?, *Impact* (12 September 2018). Available at: https://impact.chartered.college/article/mathematics-in-early-years/.

4 T. Nunes and P. Bryant, *Key Understandings in Mathematics Learning. Paper 2: Understanding Whole Numbers* (London: Nuffield Foundation, 2009), p. 4.

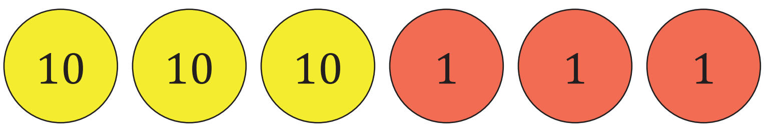
too many counters for it to be worth it. What is much better for numbers this large is to start to move towards the one-to-many relationship by grouping counters, blocks and so on into groups, usually of 10, to support the beginnings of place value. For example, if using counters these can be arranged in a 10s frame:



The frame can then act as a unit of counting, so that something like the number 33 can be seen as three of these, plus three further counters – for example:

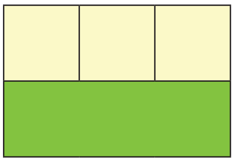


These can help make sense of numbers like 33 as ‘three 10s plus three 1s’ whilst simultaneously allowing the continued sense of the relationship between a 10 and a 1. Eventually, once pupils are secure on the relationship between a 10 and a 1 and no longer need it to be explicitly present, the whole system can be changed so that a 10 is a single object, which is what happens with place value counters:

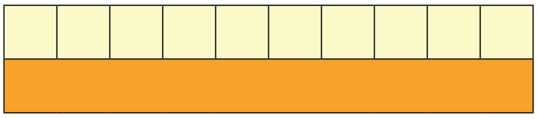


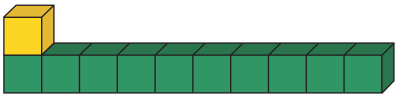
Here, the relationship between a 1 and a 10 is not visible, so this relies on pupils already being secure that a 10 is worth the same as ten 1s, but provided this is clear then this sort of representation can be a real gateway into arithmetic with larger integers.

All of these viewpoints are thinking about numbers as discrete objects. There is also the alternative interpretation of the concept of ‘number’, namely as continuous quantities. For this, useful manipulatives are Cuisenaire and Dienes. With these representations, 1 can be seen simultaneously as the number of objects but also as the length of the objects. So, three blocks represents 3 both by the number of blocks but also by the length of what is created when these blocks are lined up. Other rods with the same length can then be created to represent that number. For example, in the picture below the green rod represents 3 as it is the same length as a line of three connected 1s.



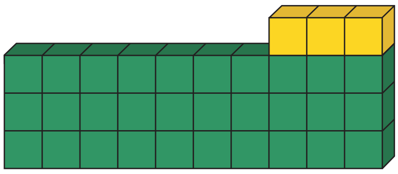
Both Cuisenaire and Dienes then have a rod that is as long as 10 of its unit cubes, allowing the same bridge into place value:



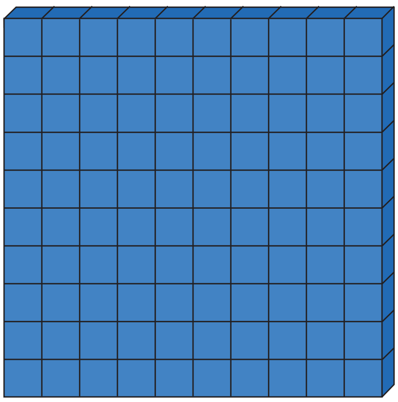


so that larger numbers can be represented continuously as either length:

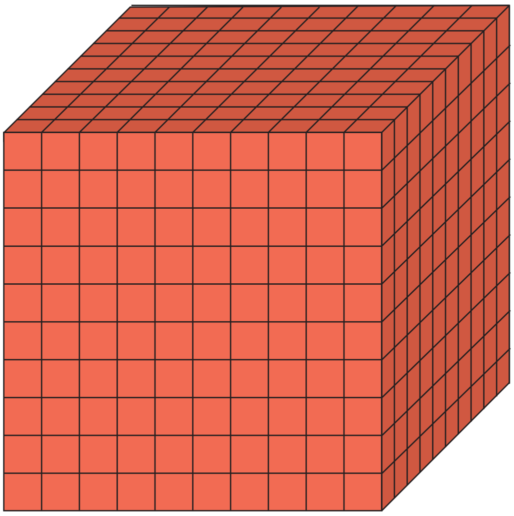
or area/volume:



This area view has very strong links to multiplication and is also what allows the representation of more than two powers of 10 using Dienes blocks, as there are also blocks that can be used to represent ten 10s (100):



and then, by generalising further into three dimensions, uses volume to represent ten 100s (1000):



To make this bridge, questions can be asked such as, 'Why is this 3?' (in reference to three blocks) and then, when pupils refer to the number of blocks, showing them that an alternative view is to consider them connected together to create a length. Pupils can then be asked to 'find' different values in the rods. Sentences such as, 'If the length of the white rod is worth one (an important part as the length of the white rod will be changed later), show me the rod whose length is worth …' can be a good way to both secure understanding of number as continuous quantity and also to support gaining familiarity with the concrete resources. This can then later become, 'If the length of the white rod is worth one, show me …' which can allow pupils to use area and volume to show larger values as place value is introduced. Obviously, place value is an important subconcept of number that pupils must secure if they are going to deal with larger numbers. It will be important to make sure pupils recognise the relationship between different place values and multiplication/division by 10.

Subconcept: place value

Prerequisites: Natural numbers.

Linked concepts: Multiplication, division, fractions, decimals, negatives, surds, algebra, indices, equality/equivalence.

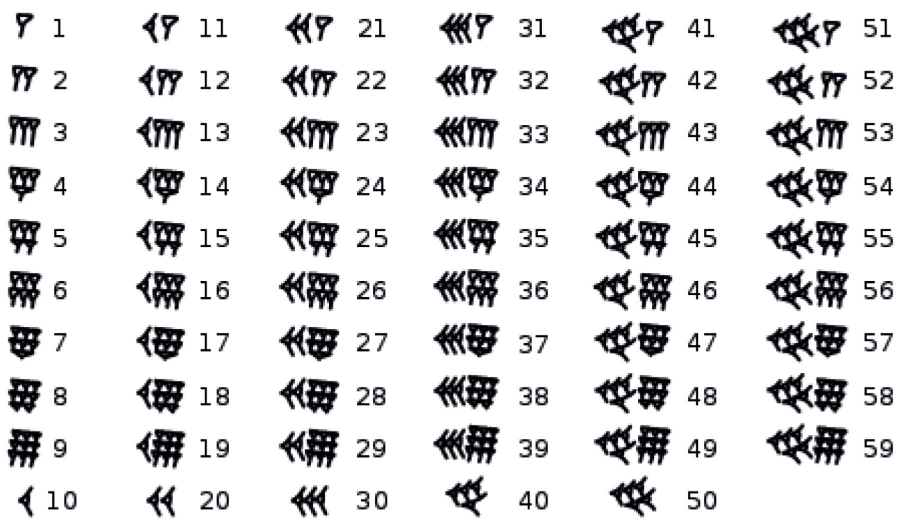
Understanding place value rests in recognising the truth in the following question:

What is ρ?

Now, if you put that in front of many English people, they might think you have written a funny-shaped letter 'p'. However, in Greece it would be recognised as the letter 'ro'. If you showed it to a scientist, they might see it as the symbol they use for density (depending on the branch of science). A statistician might think you meant a Spearman's rank correlation coefficient.

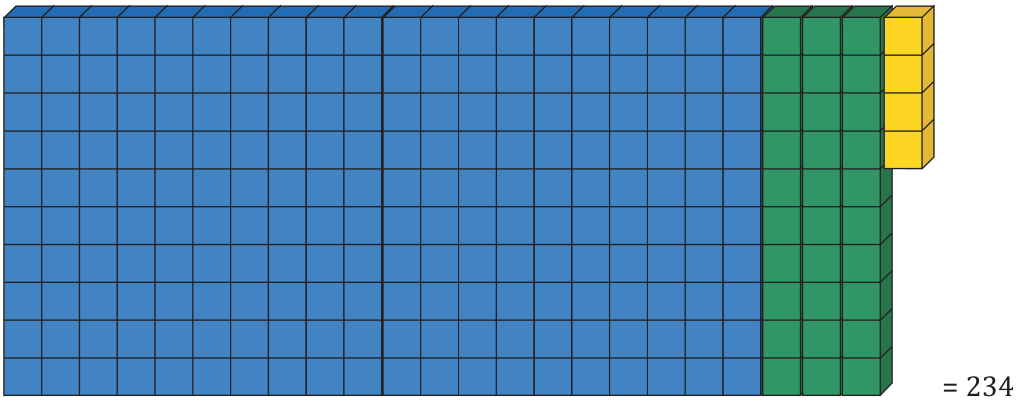
The point is that ρ is a symbol and nothing more. The meaning it conveys depends entirely on what we have learnt to associate with it. The same is true of the symbols we use to represent numbers – in our base 10 number system the symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 (usually called digits). These digits can be combined to create **numerals**; 234 is a numeral in that it represents a number in a symbolic form. We are perhaps not used to thinking of things like 234 as numerals, but the idea is no different to writing CCXXXIV – both are numerals designed to represent the same number. The number itself is something abstract – we can represent it symbolically or using different models, but its true existence is in the realms of pure thought. We have already seen that we can choose *how* we think about numbers – they can be thought of as discrete objects, continuous measures or perhaps in other ways not yet thought of (at least by this author!).

The idea of a place value system then, is that it means we can build numerals to represent larger numbers out of relatively few digit symbols. How many digit symbols depends on the **base** of the place value system – we are probably most used to using a denary (base 10) number system, but there are lots of examples of other bases both historical and modern. Computer base language is binary, which is a base 2 number system using only the digits 0 and 1. Colour codings in computers use a hexadecimal number system, which uses the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F. In ancient Mesopotamia they used a base 60 (sexagesimal) number system that had a sub-base of 10. What this means is there was a new symbol for 10, and 11 was shown as '10 and 1', 21 was shown as 'two 10s and 1' and so on up to 59 being shown as 'five 10s and nine 1s', and then 60 uses the same symbol as 1, but in the next place:



Source: https://en.wikipedia.org/wiki/Sexagesimal. CC BY-SA 4.0

Now, of course, we may not want to go into this level of detail with young or novice learners when it comes to place value. The important thing for learners to understand is that when writing numbers symbolically, the place that the digit occupies within the numeral is designed to convey the meaning about the size of the number that the digit represents. We can support early understanding of this by using concrete or visual representations alongside the symbolic. Dienes blocks are designed specifically for representing numbers in base 10:

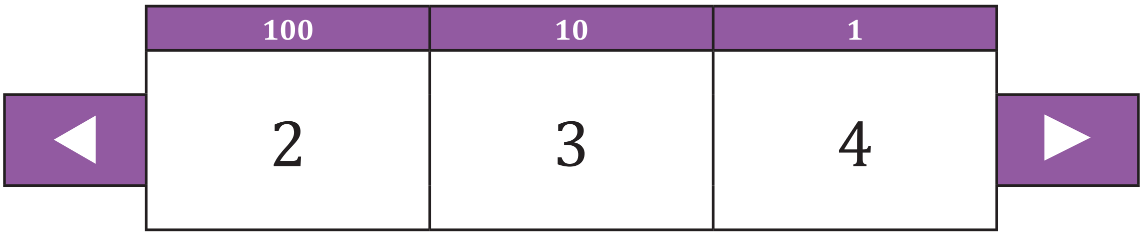


We can draw further attention to the place value by using a Gattegno chart or place value chart alongside the concrete/visual:

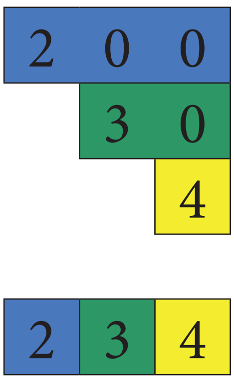
Gattegno chart

100	200	300	400	500	600	700	800	900
10	20	30	40	50	60	70	80	90
1	2	3	4	5	6	7	8	9

Place value chart



A nice physical version of the place value chart is to use place value cards alongside (and eventually to replace) the Dienes:



Conceptual Maths

Empowers teachers to support students on a comprehensive and coherent journey through school mathematics, showcasing the best models, metaphors and representations and providing excellent examples, explanations and exercises that can be used across the curriculum.

Concepts are at the heart of the study of mathematics. They are the ideas that remain constant whenever they are encountered, but which combine and build upon each other to create the mathematical universe. It is the structure of each concept that gives rise to the procedures that are used in calculation and problem-solving – and, by learning about these structures, a learner can make sense of how different processes work and use them flexibly as need demands.

In his first book, *Visible Maths*, Peter Mattock focused on the use of representations and manipulatives as images and tools and how this can provide a window into some of these mathematical structures. His aim in *Conceptual Maths* is to go deeper, beyond the procedures, and to shed greater light on the structures of the subject's different concepts.

Peter delves into a broad range of concepts: number; addition and subtraction; multiplication and multiples; division and factors; proportionality; functionality; measures; accuracy; probability; shape and transformation; and vectors, among many others. In so doing, he equips teachers with the confidence and practical know-how to help learners assimilate knowledge of mathematical concepts into their schema and take their learning to the next level.

Suitable for teachers of maths in primary, secondary and post-16 settings

Peter Mattock focuses on the development of robust conceptual knowledge in school mathematics – where do core mathematical ideas come from and where might they lead?

Anne Watson, Emeritus Professor of Mathematics Education, University of Oxford

A focus on concepts over processes makes *Conceptual Maths* a valuable read for anyone teaching mathematics.

Jemma Sherwood, Senior Lead Practitioner of Mathematics at Ormiston Academies Trust and author of mathematics education books

This book will become a central reference for anyone teaching mathematics – not just for teaching but for the pure joy of understanding the structures of mathematics from various new viewpoints.

Atul Rana, international online tutor and host of #MathsChatLive

Conceptual Maths offers that lightbulb moment for any teacher wanting to make sense of the key mathematical concepts they teach.

Dave Tushingham, Lead Practitioner, Blaise High School and co-founder of the GLT & Friends Book Club

I cannot wait to share this book with colleagues from all areas of mathematics teaching.

Lisa Coe, Primary Maths Lead for Inspiration Trust

Conceptual Maths is a superbly detailed exploration of the structure of key maths concepts which will be useful for anyone who is interested in the teaching and learning of school-level mathematics.

Charlotte Hawthorne, Lead Practitioner, St John Fisher Catholic College

Peter Mattock has been teaching maths for over 15 years. He is a specialist leader of education (SLE) and an accredited secondary maths professional development lead, who regularly presents at conferences across the country. Peter also develops teaching for mastery in the secondary school classroom, having been part of the first cohort of specialists trained in mastery approaches by the National Centre for Excellence in the Teaching of Mathematics (NCETM). **@MrMattock**

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