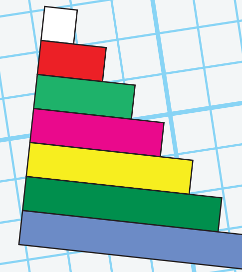
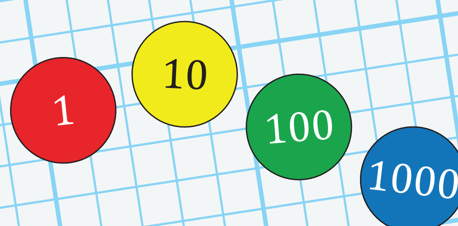
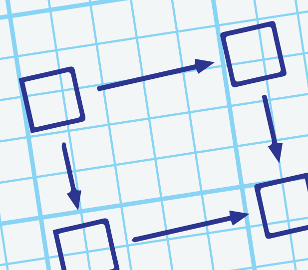
Peter Mattock



Visible Maths

Using representations and structure to enhance mathematics teaching in schools





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Acknowledgements

This book is dedicated to:

All the maths teachers and education professionals I have worked with; you have all shaped my practice in some way that has led me to this point.

All of the maths teachers and education professionals I converse with on Twitter; you are a constant source of inspiration that challenges me to keep getting better. I am not going to try and list you all (we all know how that goes, Julia @TessMaths), so just know that I value our conversations and debates.

My mum, Lesley Mattock, who has been one of the greatest sources of support to me throughout my life and without whose help I would never have reached this point.

But mostly this book is dedicated to my loving partner Rowan. Her understanding and support throughout my career and throughout this project has been beyond what any man could hope for. She has put up with so much whilst I have worked to get to where I am in my career, and even more as I balanced leading a mathematics department with writing this book. Without her love this book would never have been written.

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Jan Parry, Dr Mark McCourt, Professor Anne Watson, Professor John Mason, Pete Griffin and Steve Lomax, all of whom between them have taught me the importance of actually having a personal pedagogy and opened my eyes to the importance of ensuring that pedagogy is well informed, as well as influencing my knowledge of representation, structure, variation theory and a host of other important approaches in an array (no pun intended!) of different ways.

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Contents

Introduction

There is a great mathematics story that I was told in a lecture at university. It involves two donkeys and a fly. The problem goes that two donkeys are 100 metres apart and walking directly towards each other at 1 metre per second. A fly starts on the nose of the first donkey and buzzes between the noses of the two donkeys at 10 metres per second. The question is, how long before the fly is crushed between the two donkeys?

One of the ways to solve this problem is summing an infinite series (i.e. summing the terms of a sequence that continues forever). On its way to the second donkey the fly is travelling for $\frac{100}{11}$ seconds, then on the way back $\frac{900}{121}$ seconds, then another $\frac{8100}{1331}$ seconds, and so on. The *n*th term of the geometric series is given by $\frac{100}{11} \times (\frac{9}{11})^{n-1}$ and so the sum to infinity of the series is $\frac{100/11}{1-9/11} = \frac{100/11}{2/11} = \frac{100}{2} = 50$ seconds.

The other way to solve the problem is to ignore the fly completely. Each donkey is walking at 1 metre per second. This means that they will meet halfway at 50 metres. If they travel 50 metres at 1 metre per second it will take 50 seconds.

The story goes that a group of university students were told that a natural mathematician would automatically try to solve the problem using an infinite series and a natural physicist would solve it using the simpler approach. The problem therefore sorted mathematicians from physicists: if a student were able to solve it in a few seconds they were a physicist and if not they were a mathematician. The undergraduates were posing the problem to various students passing through the university library when the famous mathematician Leonhard Euler walked by. They presented the problem to Euler and were amazed when he answered the problem within a few seconds, as they had automatically expected him to begin considering the infinite series. When one of the students explained that a natural mathematician would have begun by forming the infinite series for the motion of the fly, Euler replied, ‘But that is what I did …’

A very similar story exists about the eminent mathematician and computer scientist John von Neumann and trains, which makes me suspect that this is at best a parable about Euler and at worst a case of Chinese whispers. However, the point of the story is not to show how good at mathematics Euler (or von Neumann) was, but instead to show that sometimes in mathematics the way you think about the calculation or problem you are solving has a great impact on how simple the problem is or how much sense it makes. Only the best A level mathematics students would be able to form the infinite series necessary to solve the problem, whereas most early secondary school pupils would be able to work out the simpler solution.

The importance of having different ways to view even the most simple mathematics, in order to build up to more complicated ideas, cannot be overstated. Some ways of thinking about numbers make some truths self-evident, whilst simultaneously obscuring others. In the same way in physics that it is sometimes better to view elementary matter as particles and at other times as waves, so in mathematics it is sometimes better to view numbers as **discrete** and at other times as **continuous**, as counters or bars, as tallies or **vectors**. Crucially for teachers, being explicit about how we are thinking about numbers and operations, and encouraging pupils to think about them in different ways, can add real power to their learning.

Much has been made of the effectiveness of metacognition in raising the attainment of pupils. For example, John Hattie lists metacognitive strategies as having an effect size of 0.6 in the most recent list of factors influencing student achievement.[*] Ofsted also recognises the importance of the use of manipulatives and representations to support flexibility in pupil thinking. In their *Mathematics: Made to Measure* report from 2012, it is noted that schools should choose 'teaching approaches and activities that foster pupils' deeper understanding, including through the use of practical resources, [and] visual images'.[†] In *Improving Mathematics in Key Stages Two and Three*, the Education Endowment Foundation lists 'Use manipulatives and representations' as one of its key recommendations.[‡] It is therefore important that we give the pupils the tools they need in order to think about the mathematics they are working with in different ways.

The use of representations and structure is also an important part of teaching for mastery approaches. The National Centre for Excellence in the Teaching of Mathematics (NCETM) lists representation and structure as one of the 'Five Big Ideas' in teaching for mastery.[§] The NCETM make clear that using appropriate representations in lessons can help to expose the mathematical structure being taught, allowing pupils to make connections between and across different areas of maths. They also emphasise that the aim in using these representations is that pupils will eventually understand enough about the structure such that they do not need to rely on the representation any more. This is often summarised as employing a concrete-pictorial-abstract (or CPA) approach to teaching mathematics.

Recently re-popularised in the UK following the focus on teaching approaches imported from places such as Shanghai and Singapore, the CPA approach actually has at least some of its roots in the 1982 Cockcroft Report, which reviewed the teaching

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* See https://www.visiblelearningplus.com/sites/default/files/250%20Influences.pdf.

† Ofsted, *Mathematics: Made to Measure* (May 2012). Ref: 110159. Available at: https://www.gov.uk/government/publications/mathematics-made-to-measure, p. 10.

‡ See P. Henderson, J. Hodgen, C. Foster and D. Kuchemann, *Improving Mathematics in Key Stages Two and Three: Guidance Report* (London: Education Endowment Foundation, 2017). Available at: https://educationendowmentfoundation.org.uk/public/files/Publications/Campaigns/Maths/KS2_KS3_Maths_Guidance_2017.pdf, pp. 10–13.

§ See https://www.ncetm.org.uk/resources/50042.

of maths in England and Wales.* The Cockcroft Report advocated (among many other things) the need to allow pupils the opportunity of practical exploration with concrete materials before moving towards abstract thinking.

There are several studies on the use of manipulatives across the age and ability range, with most showing that mathematics achievement is increased through the long-term use of concrete materials. The most comprehensive of these is Sowell's 'Effects of Manipulative Materials in Mathematics Instruction', a meta-analysis of 60 individual studies designed to determine the effectiveness of mathematics instruction with manipulative materials.† Those surveyed ranged in age from pre-school children to college-age adults who were studying a variety of mathematics topics. Sowell found that 'mathematics achievement is increased through the long-term use of concrete instructional materials and that students' attitudes toward mathematics are improved when they have instruction with concrete materials provided by teachers knowledgeable about their use'.‡

The aim of this book is to explore some of the different concrete materials available to teachers and pupils, ways of using these concrete and pictorial approaches to represent different types of numbers as discrete or continuous, how certain operations work when viewing numbers in these ways, and how these various representations can help to support the understanding of different concepts in mathematics. The book will look at the strengths of each representation, as well as the flaws, so that both primary and secondary school teachers of mathematics can make informed judgements about which representations will benefit their pupils. I will draw on my own experience of using the representations, as well as experiences shared by others, and appropriate research in order to support teachers in understanding how these representations can be implemented in the classroom.

How to use this book

I have often noticed that one of the difficulties pupils have in acquiring new mathematical understanding is that we introduce new ways of representing or thinking about mathematics at the same time as we try to teach a new mathematical concept or skill. I will take an alternative approach here, which is to explore all of the representations first and then, once they are secure, examine how more complicated calculations and concepts can be developed.

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* W. H. Cockcroft (chair), *Mathematics Counts: Report of the Committee of Inquiry into the Teaching of Mathematics in Schools* [Cockcroft Report] (London: HMSO, 1982). Available at: http://www.educationengland.org.uk/documents/cockcroft/cockcroft1982.html.

† E. J. Sowell, Effects of Manipulative Materials in Mathematics Instruction, *Journal for Research in Mathematics Education*, 20(5) (1989), 498–505. Available at: http://www.jstor.org/stable/749423?read-now=1&seq=7#references_tab_contents.

‡ Sowell, Effects of Manipulative Materials in Mathematics Instruction, 498.

I wouldn't introduce all of these representations at once with pupils; instead I would introduce two or three. Importantly, though, I would ensure that pupils are comfortable with the representation before trying to use the representation to explore a new concept. This generally involves introducing the representation to pupils within a concept they are comfortable with, and modelling with them how the representation fits with what they already know. This then allows the teacher to develop the concept into something new, using the representation as a bridge.

As this book is aimed at teachers, Chapter 1 will set out all of the representations within the secure concept of whole numbers, and Chapter 2 will then extend these representations to include fractions and decimals. The basic operations of addition and subtraction of whole numbers will be introduced in Chapter 3, followed by multiplication and division of whole numbers in Chapter 4, and powers and roots of whole numbers in Chapter 5. Chapter 6 then explores these ideas as applied to fractions and decimals. Chapter 7 examines the use of representations to illustrate the fundamental laws of arithmetic, and then in Chapter 8 we look at how these combine to define the correct order of operations in calculations involving multiple operations. Chapter 9 covers the concepts of accuracy, including **rounding**, significant figures and bounds, before we move on to **irrational numbers** in Chapter 10.

Chapter 11 sees the introduction of different representations applied to algebra, after which we progress to manipulating algebraic expressions by simplifying expressions (Chapter 12), multiplying expressions (Chapter 13) and expanding and factorising expressions (Chapter 14). In Chapter 15 we look at how representations can support with illustrating the solutions of **equations**, and then Chapter 16 examines some particular algebraic manipulations not covered in Chapters 12 to 14 – in particular, the difference of two squares and completing the square. Finally, Chapter 17 seeks to answer some of the questions about the use of representations in the classroom that may arise from the reading of the book.

You will notice while reading the book that some key mathematical terms are presented in **bold** – for your convenience these terms are defined in a glossary, found at the back of the book.

The fact that the book spans almost the complete breadth of primary and secondary school mathematics might make some question the usefulness of covering everything in one text. One reason I have chosen to do so is that I feel it is important that teachers understand not just the stage they are teaching, but also how this builds on what has been taught before and how this is built on in the stages after. This ensures that teachers see how what they are teaching fits into the wider pupil journey, and can support pupils no matter where they are along the way. Pupils will enter and leave stages of schooling at many different points, and just because we might teach in a secondary school doesn't mean we won't need to support pupils who haven't secured concepts from primary school, or similarly that teachers in primary schools won't need to provide depth in a topic by allowing pupils to explore a concept to a point that would normally be taught in secondary school. In this, all-through (3–18) schools have an advantage as they can design their curriculum to build all the way through the school. Those working in separate primary and secondary schools, or other school models, must use strong transition links to make this happen. So, for primary school teachers, this book showcases the mathematics you will teach and show you how it extends into secondary school. For secondary teachers, this book will provide some insight into approaches that might be used in feeder primaries and how you can develop them in secondary school.

I hope this book will support teachers in choosing suitable representations for use in their classrooms by making them much more secure in their own understanding of the strengths and weaknesses of each representation, but also, importantly, of how the representations highlight different interpretations of the concepts we explore with pupils. Some of the examples in the book will be suitable for direct use with pupils in the classroom, whilst some will be of more benefit to teachers in developing their own understanding. Pupils will very often need more than the one or two examples illustrated at each stage; in many cases, they will need to experience careful modelling with multiple examples as well as have the opportunity to explore concepts with the different manipulatives and representations provided. Only in this way will pupils eventually move beyond the representations.

The true aim of this book is for teachers to feel sufficiently confident in the use of the representations that they can explain enough about the underlying structures of the different concepts so that pupils no longer need to rely on the representations to see these structures. This is an important end goal for teachers to keep in mind – pupils should be aiming to move beyond the representation. Representations are tools that provide a window into the underlying structure of a concept. They are a window that pupils can keep coming back to look into, but they are not a window they should continually have to stare through. There is a danger that representations become another procedure that pupils have to remember and apply without understanding; this must be avoided at all costs if pupils are going to work towards mastery of mathematical concepts. This is why multiple representations are used for each concept, and why the literature makes clear the need for multiple representations to ensure pupils have a range of ways of thinking about concepts.

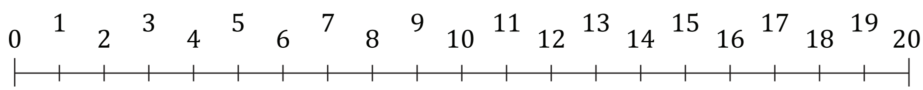
Chapter 2

Representing fractions and decimals

Now that we have a clear understanding of the different representations that will be useful in exploring different facets of numbers and numerical relationships, it is time to begin developing the number system beyond the integers.

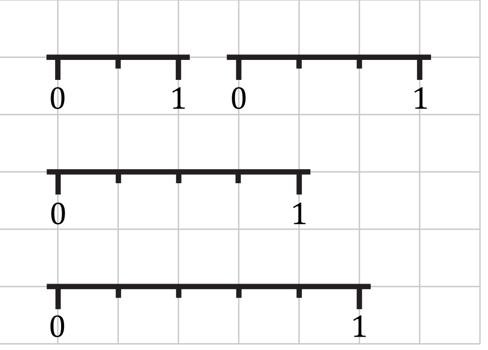
At this point, it is important to draw a distinction between representing a *number* and developing a *concept*. In the counters representation in Chapter 1, we saw that it was possible to represent negative integers using counters of different colours (or suitably labelled counters). However, this requires pupils to have developed the concept of negative integers. Using counters to represent positive integers doesn’t motivate the need for negative integers; it simply allows us to manipulate them once we already know they are there. It is important that any representations we use make the introduction of the concept a natural consequence.

The number line is a good example. When representing positive integers (and 0) on a number line like the one below, it is quite natural to consider questions like, ‘Is there anything to the left of 0?’ and so develop the concept of negative integers.



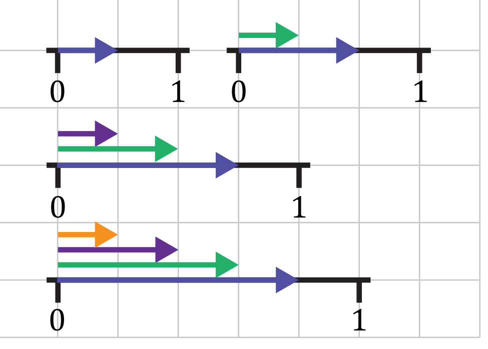
Another instinctive question to ask is, ‘What is in the space between 0 and 1?’ This allows teachers to introduce the concept of fractions and develop learners’ understanding of numbers that are less than 1. In England, according to the national curriculum, this process begins as early as Year 1 (when pupils are 5 or 6) and continues throughout primary school.

Whilst the number line on page 21 is useful for motivating a discussion about what is between 0 and 1, it is perhaps not the best representation to answer the question. A closer look at the space between 0 and 1 would appear to be in order:

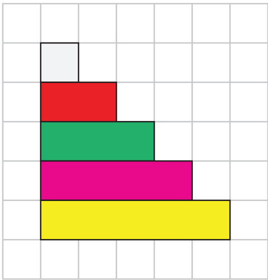


Each of these separate number lines prompts the need for different fractions by breaking the space between 0 and 1 into a different number of equal sized parts. In the first case, the number line shows $\frac{1}{2}$, in the second $\frac{1}{3}$ and $\frac{2}{3}$, in the third $\frac{1}{4}$, $\frac{2}{4}$ and $\frac{3}{4}$, and in the fourth $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$ and $\frac{4}{5}$. Other properties of fractions can also be discerned – for example, that $\frac{1}{2} = \frac{2}{4}$ or that $\frac{5}{5} = 1$.

Whether using the positions on a number line or the vector representation, the same 'zooming' of the number line is required:

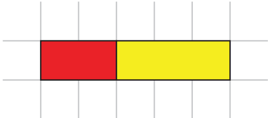


A similar approach is used when representing fractions using a bar model or physically using Cuisenaire rods:

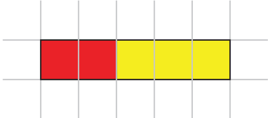


When representing integers, the white block would normally be valued as 1, the red 2, the green 3 and so on (although if we wish we can say that the white block is 2, so the red is 4, the green 6, etc.). An obvious question to ask is, 'What if the red block is 1?' Similarly, the green, pink, yellow or any other block can also be considered as 1, leading to the other blocks representing different fractions.

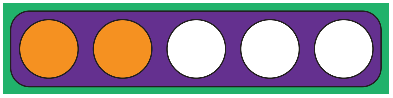
Whilst the Cuisenaire rods can be placed next to each other (as in the diagram above), a pictorial representation for modelling fractions as bars would normally have overlapping bars:

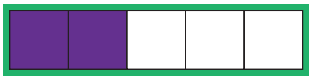


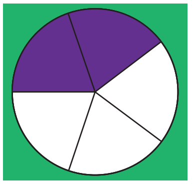
Or even have the squares filled in:



This leads to the typical ‘part-shaded’ representation of fractions, such as these diagrams:





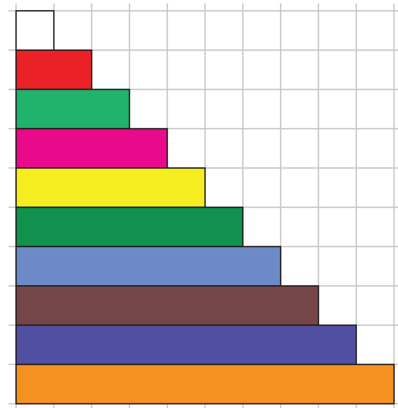


The part-shaded representation is very particular to fractions, which is why it didn’t appear in the section on counters in Chapter 1. I realise that this breaks my self-imposed approach of introducing representations before they are needed. However, I justify this in two ways: first, I cannot really see how this representation would be introduced in the classroom before this point; and, second, it develops directly from the bar model and so the representation itself isn’t a big conceptual leap.

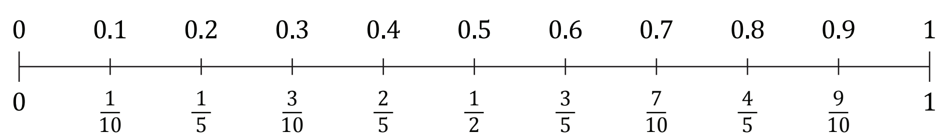
I find this an interesting representation from the point of view of the mathematics classroom. It is typically the first representation that is used with pupils in primary schools when developing the concept of fractions, and I suspect that this may be part of the reason why many pupils go on to find fractions difficult. The representation does highlight a valid interpretation of fractions, the relative counter, which is a way of counting how much is being selected out of a larger total. In the diagrams above, this would be ‘2 selected out of 5 in total’ or simply ‘2 out of 5’. The idea of a relative counter links to the idea of fractions being part of one whole, which both the bar model and number line are also designed to show. However, the language that both pupils and instructors tend to use around fractions suggests that this link is often not made explicit.

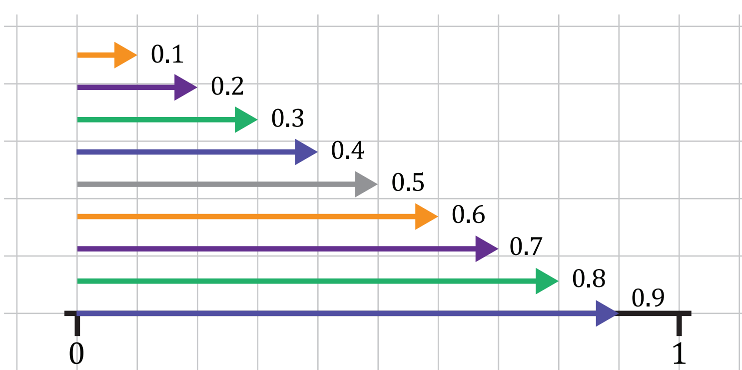
In my experience, many pupils internalise an early view of fractions as two separate values rather than a number in its own right (i.e. ‘2 out of 5’ rather than ‘two-fifths’). An unintended consequence of this early internalisation is the difficulty that some older pupils have in calculating with and manipulating fractions. As part of developing operations and reasoning with fractions, it is important that pupils understand fractions as being single quantities. Cuisenaire rods, bar models and placing fractions on a number line seem to develop this understanding much more clearly, and so I would advocate waiting to introduce the relative counter representations of fractions and working exclusively with bar models and number lines until pupils’ understanding of fractions as single values is well embedded. The 2014 English national curriculum suggests that in Year 1 pupils should be taught to represent numbers on a number line and also be familiar with simple fractions such as halves and quarters, so there would appear to be scope here to combine these to develop the interpretation of fractions as single values rather than as separate **numerators** and **denominators**.

A particular case that arises from working with Cuisenaire rods is using the ‘10’ bar to represent 1, leading to the family of fractions $\frac{1}{10}$, $\frac{2}{10}$, $\frac{3}{10}$ and so on:

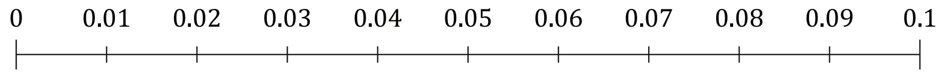


The diagram above shows the link between fractions and decimals (normally introduced in Year 3): the white bar can be seen as $\frac{1}{10}$ or 0.1, the red bar as $\frac{2}{10}$ or 0.2 and so on. The bars/rods or the number line/vector representation can be used to develop the concept of fractions or decimals as parts of a whole as a natural result of subdividing a single unit into smaller chunks of varying sizes:

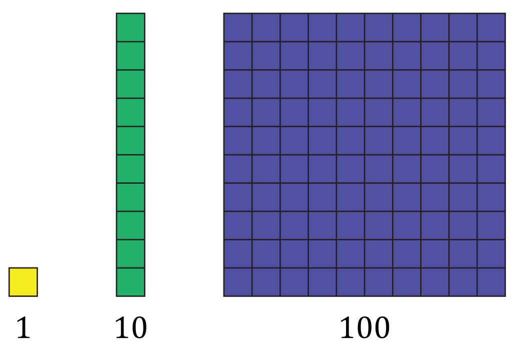




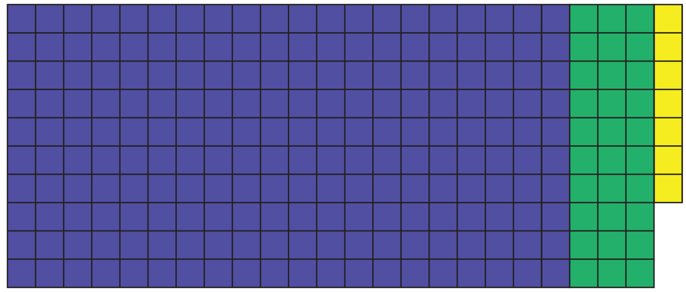
Two properties of the bars/rods and number lines/vectors make them unsuitable for the deeper development of certain concepts around fractions/decimals. The first of these is that it is much more difficult to proceed beyond a single decimal place. Unless we draw a very long number line, we are going to struggle to actually model the meaning of decimals such as 0.02, particularly if we want to retain the relative size compared to a single unit. Whilst it is possible to zoom in further on the number line (as in the diagram below) or ask the question, 'What if the orange bar becomes 0.1?' I would suggest that a consistent reference back to 1 is beneficial as these concepts begin to be formed and explored.



The representation that allows us to explore multiple powers of 10 whilst maintaining this reference back to the size of one whole is, of course, base ten blocks. When looking at integers we have the following progression:

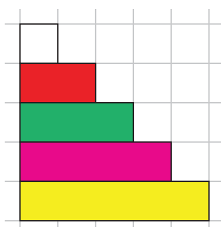


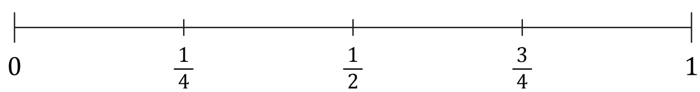
A natural question to ask is, 'What if the blue square represents 1?' This allows us to extend decimals right down to the hundredths layer (or even to the thousandths layer if using the cube as 1 with the concrete resource), which means pupils can use the different sized blocks to explore patterns like 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09, 0.1 or to show that 0.4 is greater than 0.36, which can be a major stumbling block as pupils develop an understanding of place value for numbers between 0 and 1.



2.37

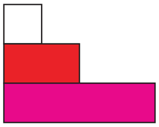
The second issue around using bars, number lines and base ten blocks is that we can only represent a certain family of fractions/decimals in the same diagram. For example, if using the yellow bar below to represent 1, all the other bars can only represent fractions in the family of fifths, or on a number line with four partitions between 0 and 1, we can only represent fractions in the family of quarters.





We can begin to solve this problem by exploring the idea of equivalence. In the number line above, we can see that two quarters is halfway along the line between 0 and 1. Both the number line and the bar model can be used to develop the concept of equivalent fractions. Teachers will need to decide whether the introduction of equivalent fractions should follow on immediately from the introduction of fractions or whether to wait a while before we revisit equivalence. What is certain is that the equivalence of fractions has to be fully understood before we move on to concepts such as addition and subtraction with fractions. The English national curriculum suggests that equivalence of fractions should be first taught in Year 2, with pupils expected to recognise the equivalence of $\frac{1}{2}$ and $\frac{2}{4}$.* With bars this may go something like this:

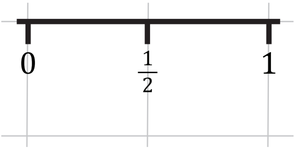
If the pink bar is 1, the red bar is $\frac{1}{2}$.



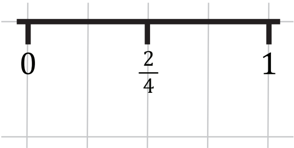
If the pink bar is 1, the white bar is $\frac{1}{4}$ – which means the red bar is $\frac{2}{4}$.

So $\frac{2}{4} = \frac{1}{2}$.

On a number line this may look something like this:



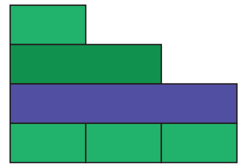
The zoomed number line above shows $\frac{1}{2}$.



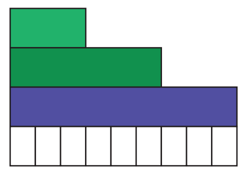
This second zoomed number line shows $\frac{2}{4} = \frac{1}{2}$.

* Department for Education, National Curriculum in England: Mathematics Programmes of Study. Statutory Guidance (July 2014). Available at: https://www.gov.uk/government/publications/national-curriculum-in-england-mathematics-programmes-of-study/national-curriculum-in-england-mathematics-programmes-of-study. All future references to national curriculum guidelines (from Key Stage 1 onwards) refer to this publication.

This doesn't only work with halves. The diagrams below use bars to show the same idea with thirds and ninths:



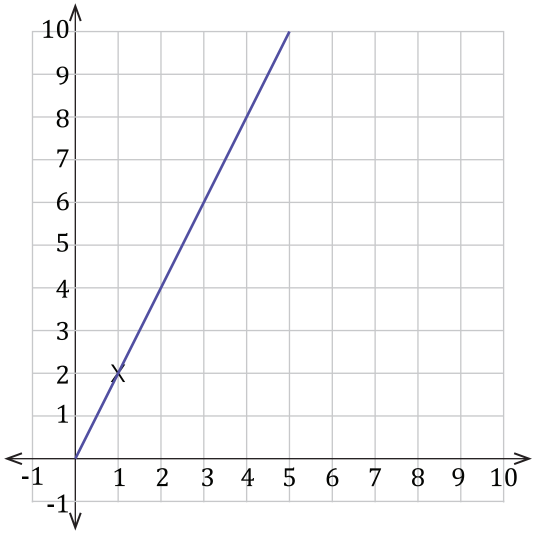
If the blue bar is 1, the light green bar is $\frac{1}{3}$ and the dark green bar is $\frac{2}{3}$.

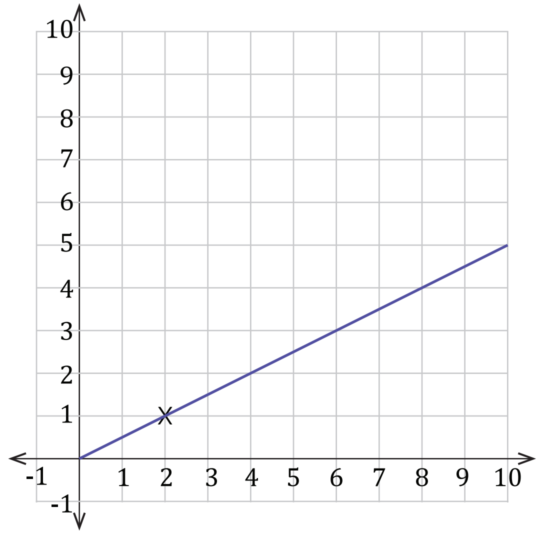


If the blue bar is 1, the white bar is $\frac{1}{9}$, so the light green bar is $\frac{3}{9}$ and the dark green bar is $\frac{6}{9}$. So $\frac{3}{9} = \frac{1}{3}$ and $\frac{6}{9} = \frac{2}{3}$.

If we want to represent many different families of fractions in the same diagram, then we will need a representation that is designed to show multiplicative relationships – namely the ordered-pair graph.

The ordered-pair graph could be seen as overly complicated for integers as it requires the use of two values to represent what is, in essence, just a single quantity – for example, the coordinate (1, 2) to represent 2. But when used as a tool to conceptualise and represent **rational numbers** it is supremely powerful. The conceptualisation comes from questions such as, 'If the coordinate (1, 2), and therefore the straight line from (0, 0) to (1, 2), represents the number 2, what does the coordinate (2, 1) represent?'





This approach doesn't necessarily capture the 'part of a whole' view of fractions; instead, it is much more like the ancient mathematicians who viewed fractions more as the ratio of two integer values rather than numbers in their own right. Indeed, when Pythagoras exclaimed 'All is number', he was referring to ratios between integers that he believed governed everything from the Music of the Spheres (the movement of the stars and planets) to actual musical harmony (which he showed was actually the case). The famous story of the drowning of Hippasus of Metapontum was a direct result of this belief.

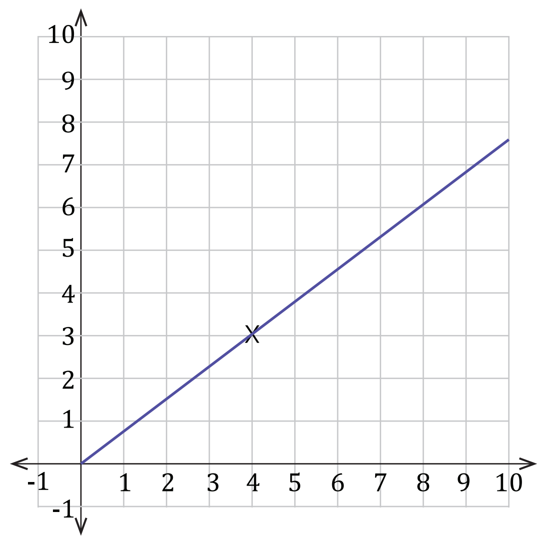
Hippasus was a philosopher from around the year 500 BC. He was an early follower of Pythagoras and part of the Pythagorean Brotherhood, the group of philosophers who worshipped number and believed that the root of all reality lies in mathematical truth. The story goes that Hippasus was working with another of Pythagoras' famous results, Pythagoras' theorem. Hippasus knew that the values of 1, 1 and $\sqrt{2}$ satisfied Pythagoras' theorem, but he was trying to find the integer ratio (fraction) that was equal to $\sqrt{2}$. It dawned on Hippasus that the reason he was unable to find the correct ratio was that it didn't exist. At this point the most popular versions of the story diverge. The first version I came across (in Simon Singh's *Fermat's Last Theorem**) was that Hippasus, rightly proud of this discovery, shared it with his mentor Pythagoras. Unfortunately, it didn't fit with Pythagoras' view of the universe being governed by integer ratios. When Pythagoras couldn't refute Hippasus' argument through logic, he resorted to drowning his student rather than admit he was wrong. The second version is that Hippasus shared his discovery that $\sqrt{2}$ is irrational outside of the

* S. Singh, *Fermat's Last Theorem* (London: Fourth Estate, 1997).

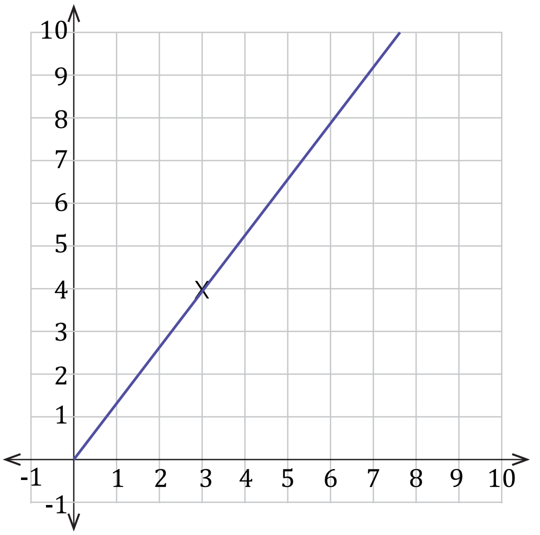
Pythagorean Brotherhood, which was strictly forbidden, and for breaking his solemn vow not to reveal the secrets of the sect he was drowned.

Of course, since the time of the ancient Greeks humans have discovered that not only do we have to go beyond the rational numbers to form a complete number system, but even beyond the irrational numbers like $\sqrt{2}$. We will explore the representation of values like this in Chapter 10, but for now we will return to fractions and decimals.

The ordered-pair graph extends beyond the coordinates (1, 2) to represent the number 2 and (2, 1) to represent the number $\frac{1}{2}$ to all other ratios. The coordinate (4, 3) can represent the number $\frac{3}{4}$, whilst the coordinate (3, 4) represents the number $1\frac{1}{3}$.



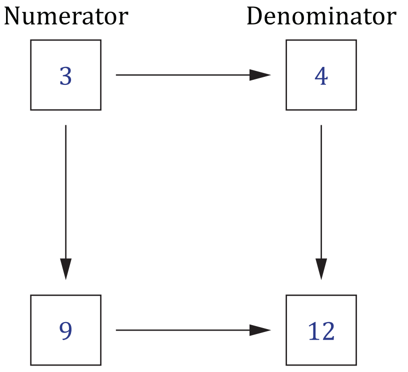
The diagram above shows $\frac{3}{4}$ on an ordered-pair graph.



This second diagram shows $1\frac{1}{3}$ on an ordered-pair graph ($\frac{4}{3}$).

One of the major strengths of this representation is that it makes very clear the relative sizes of different fractions, with steeper lines representing bigger values.

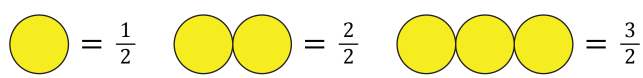
The proportion diagram can also be a useful tool for representing the multiplicative relationships within fractions. When introduced with integers in Chapter 1, this diagram was shown using the language of 'up' and 'right' to make the link between the diagram and the ordered-pair graph. When beginning to work with fractions, we change this to the standard language of 'numerator' and 'denominator':



As with the use of this representation for whole numbers, the proportion diagram doesn't so much represent the numbers themselves as it does the similarity in the relationship between the numbers – literally, '3 is to 4 as 9 is to 12'. Of course, in order to see this, pupils will need to have developed a clear concept of multiplication from being taught using different representations – at least, the basic concept of a fraction through working with the ordered-pair graph. As such, the proportion diagram isn't the best representation for introducing rational numbers or developing an early conceptual understanding of fractions. However, once this understanding is in place, this representation can be very powerful in manipulating proportional relationships, like the relationship between the numerator and denominator of a fraction.

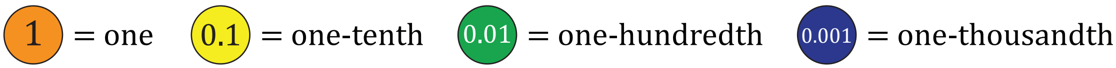
Another representation that has its uses once the concept of fractions and decimals is well developed is counters. The versatility of counters to represent different values can be useful when working with whole numbers, but they are particularly valuable when developing certain operations with fractions and decimals. For this reason alone, it is well worth finishing off our development of rational numbers by exploring how pupils understand the way these numbers look when represented as discrete objects.

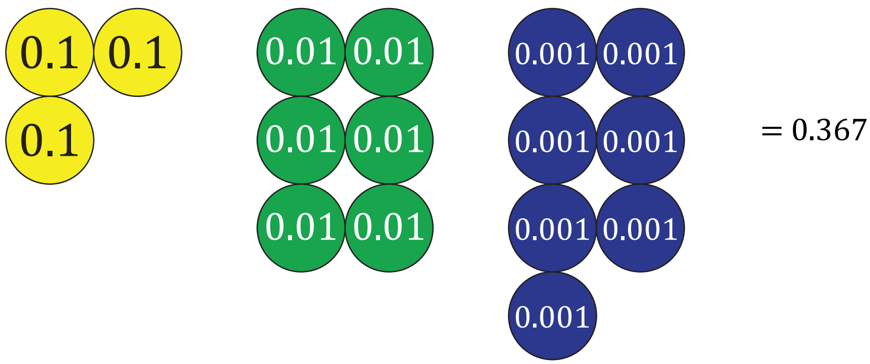
When using counters to represent fractions, it is necessary to assign a value to a counter. This could be simply taking all counters to have a certain value – in the example below this is $\frac{1}{2}$.



What this representation does not make clear are some fairly basic truths about fractions – an obvious example being the fact that two halves are equal to one whole. It is important to develop the concept of fractions in a way that makes these fundamental truths more self-evident (e.g. using the bar model, number line or ordered-pair graph) before using counters as a representation of fractions. Another conceptual difficulty here is using a 'whole' counter to represent one half or any other fraction. Pupils who can't make this leap don't really understand counting and struggle to see counting as applying to anything other than 1s. In the case above, it may be useful to explore the old half penny and other pre-decimal coins in order to help pupils develop the understanding of counters having fractional values.

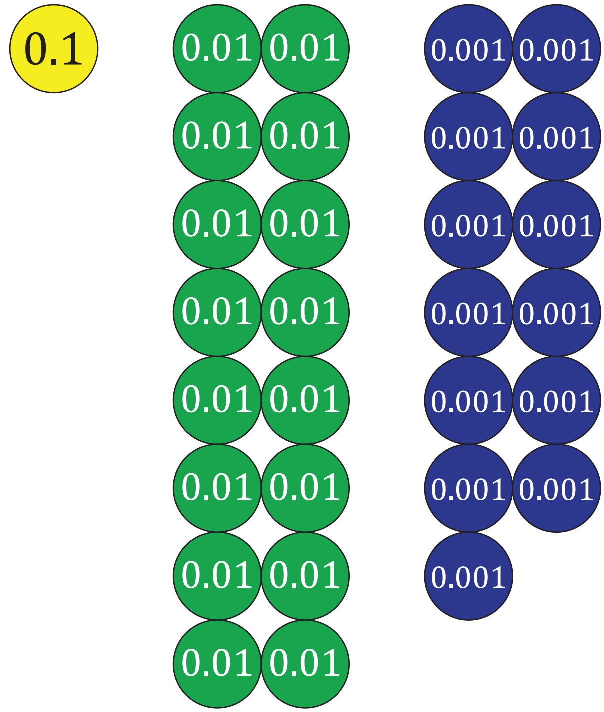
To further support the understanding of counters taking anything other than integer values, we can also revisit place value counters. There is a strong link here with pounds and pence; if £1 is 1, then 10 pence is 0.1 and a penny is 0.01. This can be shown pictorially or using plastic coins if available.



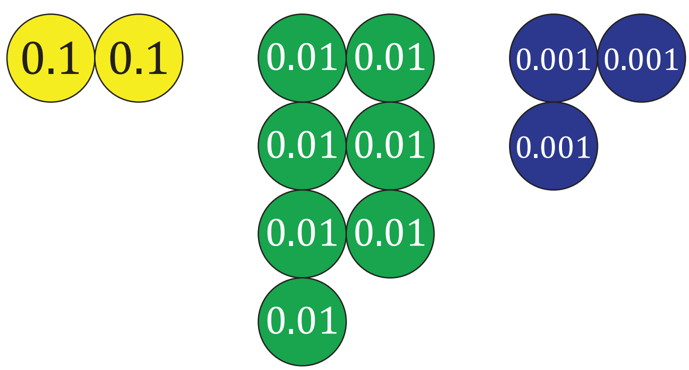


At this point it is useful is to embed, or re-embed, the concept of **exchange**. Pupils will be comfortable with this idea for whole numbers – that twelve '1' counters can be exchanged for a '10' counter and two '1' counters – but may not naturally transfer this idea to decimal place value counters or to fractional equivalence. Making the idea clear when introducing decimals through bars or number lines helps, but it is still worth getting pupils used to the idea with counters.

Some teachers are tempted to wait until exploring addition/subtraction to broach the idea of exchange. However, this means that two important concepts are introduced at the same time, which can be troublesome. Once pupils are familiar with the use of counters to represent decimals, the concept of exchange can be explored directly, in a similar way to the concept of equivalence in fractions. We might present pupils with situations like the one that follows and ask them what number is represented.

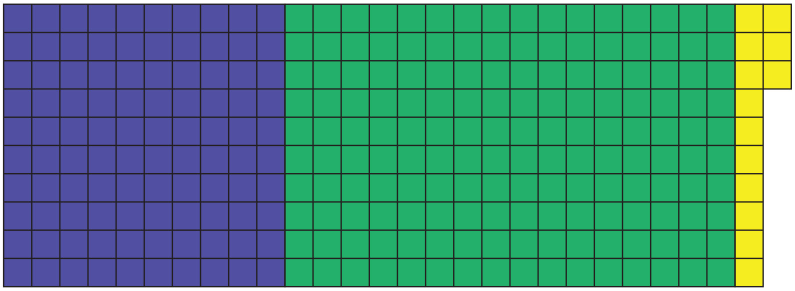


By using the understanding of place value that pupils have already developed, they should be able to see that 10 of the blue counters can be exchanged for 1 green counter, and that 10 of the green counters can be exchanged for 1 yellow counter. This leads to the result below:

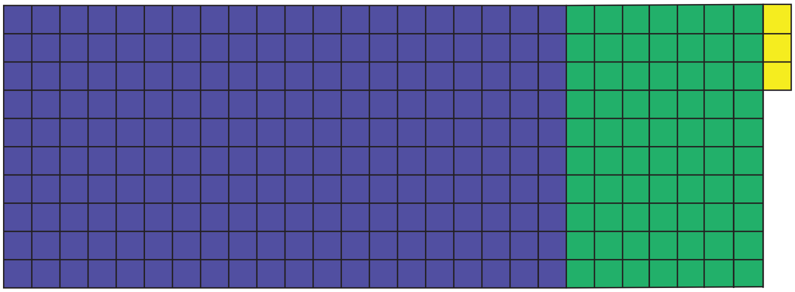


$= 0.273$

Exchange can also be explored using other representations, particularly base ten blocks, where:



exchanges for:



In fact, it would be beneficial to explore exchange with base ten blocks before counters (although, again, I wouldn't wait to explore exchange with base ten blocks before introducing counters). Ideally, the sequence would be something like:

1 Introduce decimals with base ten blocks, defining different shapes as 1.
2 Introduce the use of counters to represent different place values.
3 Explore the idea of exchange using base ten blocks, where it can be physically seen that 10 single cubes can be exchanged for a line, 10 lines for a square and so on.
4 Explore the idea of exchange with counters standing for different place values.

Of course, exchange can also be used with subtraction when we need to 'exchange down' (e.g. exchange a base ten block or counter for 10 unit blocks/counters). This situation does not naturally arise before beginning to examine subtraction, so this form of exchange should probably be left until we are teaching pupils about subtraction.

The representations outlined in the first two chapters should provide pupils with a deeper understanding of both integers and rational numbers and how these relate to each other. The next stage is to look at combining these values using different operations.

Chapter 17

Frequently asked and anticipated questions

The ideas and approaches set out in this book have never, to my knowledge, appeared in one place. There have been individual books and papers written about nearly all of these representations (although I have yet to find one about the use of vector notation to explore number and algebra or the use of ordered-pair graphs), but I have never seen them collected together and compared. I am privileged that I have occasionally been invited to speak at conferences where I have discussed some of the ideas and representations that appear in this book, but as I am sure you will appreciate, to describe them all would take a session lasting many hours, if not days.

In writing this book, I hope that teachers will acquire a greater understanding of the strengths and drawbacks of each representation (downsides are often notably lacking from texts on the individual representations), and so make informed decisions about which representations they will use to help develop their pupils' understanding. Due to the fact that these ideas have never been collected together in this way, there hasn't really been an opportunity for teachers to ask many questions about them. The answers that follow are my responses to questions that have been asked when I have talked with individuals or groups about these representations, as well as queries that I anticipate will arise from some of the ideas put forward.

1. Are these all of the manipulatives/ representations I should be using in the classroom, or that other teachers use in the classroom?

This book is by no means a complete collection of every representation that is used or can be used in the classroom. There are some notable exceptions, such as the dual number line which is used for exploring proportional relationships, as well as the multitude of representations used to develop understanding in the primary classroom. When first looking at number and operations, particularly with questions set in context, many primary teachers will use representations that engage directly with the context. For example, if an addition question is about people getting onto a bus, the teacher will model this with actual toy people and a toy bus. Some teachers use

Lego blocks. Some use Multilink cubes. I know at least one teacher who uses paper folding to teach children about fractions. I even know of teachers who use sweets to model different mathematical ideas.

What I have tried to do in this book is to capture the core representations that many of these tools devolve to – sweets and toy people can, at an appropriate point, be replaced with counters. Lego blocks can become bar models or vectors. Multilink can become bars or counters (or both). The aim is to provide teachers with both (a) a starting point from which to explore representation and structure and (b) enough understanding that there should be little in the realms of number or algebra that teachers cannot approach in a concrete or pictorial way, should they wish.

2. Should I use all of these representations with my pupils? I am not sure I have the time!

I am a strong believer in the idea that teachers are the people best placed to make decisions about the learning of their pupils, and the approaches, explanations and activities that will best support their learning. If you, as a teacher, believe that all of the representations in this book will benefit your pupils in understanding the concept(s) or process(es) you wish them to understand, then use them all. If you feel that two or three are enough to provide them with the understanding you wish them to have, or for them to develop this understanding for themselves, then choose those you feel will best support this goal.

Being aware of multiple representations and their strengths/weaknesses gives teachers a toolkit to support their pupils, so if one representation doesn't make the concept clear then you can try another, or another, until a representation is found that can illuminate the concept. You will also notice that as the book progresses, the number of active representations diminishes – in the final chapters I have limited the representations to basically counters, vectors and bars/algebra tiles.

Many of the representations have a natural shelf life beyond which they fail to properly clarify concepts. Number lines, for example, are difficult to employ with algebra, particularly when more than one variable is required. Teachers should bear this in mind when choosing representations, because we need them to support pupil understanding both now and in related concepts they might encounter in their future learning.

One thing I would like to stress is that it is definitely not necessary to introduce all of the representations at once. In the scheme of work that my department works to, we concentrate on three representations to begin with – counters, bars and number lines. Later on, we introduce the ordered-pair graphs and vectors when we work with negative integers, and then the proportion diagram, particularly when working on proportion problems. I want to reiterate that it is crucial for pupils to be comfortable with the representation prior to it being used with a new concept. So, if you want to use number lines to explore rounding, then ensure your pupils are given enough time to become familiar with number lines graduated in different units before you use them to look at rounding.

3. Pupils can't use these concrete and visual approaches in their exams, so what is the point?

Admittedly, pupils will not have access to concrete manipulatives, but I understand that exam boards are happy to credit the use of visual approaches to problem solving in GCSE exams. Many mark schemes now have lines in such as '3 × 4 or 12 seen' for awarding marks, and they are prepared to award marks for this shown on a diagram. Key Stage 2 exams only credit certain approved methods if the answer is incorrect – although, if the answer is correct then any method is allowed.

My response here is two-fold: firstly, these representations should be complicit in their own demise. The point of any representation is to make itself redundant – they allow pupils to develop the necessary understanding of the underlying structures such that they achieve the fluency to work purely symbolically (i.e. just with numerical or algebraic symbols). If teachers approach the representations as just a new process to learn, then pupils will be no better off in their understanding. However, if they are used skilfully by teachers to reveal aspects of the concept that would otherwise pass them by, then pupils can quickly move beyond the need for them.

Secondly, if the representations are used skilfully, then pupils will be much better prepared for their exams than if they see mathematics as a series of disparate processes and techniques that they are expected to learn in order to find the right answer to a question. Therefore, the point of these representations is to enable pupils to understand the structures of mathematics and the connections between them, which will ultimately result in them performing better in examinations.

4. I just use … with my pupils. What is wrong with that?

Nothing at all, provided it equips your pupils with the understanding necessary to approach the concepts that you are currently trying to support them in, and also allows them to develop an insight into the connected concepts they may meet in the future.

I first faced this question after speaking in Essex about approaching negatives using counters and zero-pairs. The questioner was talking about using temperature. My response highlighted two key problems. Firstly, temperature is not a good model for using negatives beyond basic addition and subtraction. Creating a situation where a positive temperature needs to be multiplied by a negative value is challenging, to put it lightly. I cannot envisage a scenario in the context of temperature where this would be necessary, never mind illuminating the underlying structure of multiplication and how it applies to negative integers.

The second problem is that of reach: teachers have been using temperature to teach negative numbers for years, and yet many pupils continue to make basic mistakes when it comes to calculating with negative numbers. Clearly, temperature works for some pupils, but not all. These other pupils require a different model if they are to succeed, and as teachers we should have a wealth of approaches to call on to support pupils in developing this understanding. As a model for negative values, temperature does not allow pupils to achieve the 'mastery' that Bloom talked about – where over 90% of pupils, given enough time, can secure a concept.[*] This teacher felt comfortable with the idea of temperature as a model for negatives, but it is not for us to pass on our biases to our pupils. Just because we are at ease with a model, this does not mean that it will necessarily do the job for our pupils, so it is important we have a number of approaches at our disposal.

* B. S. Bloom, Learning for Mastery, *Evaluation Comment*, 1(2) (1968). Available at: https://programs.honolulu.hawaii.edu/intranet/sites/programs.honolulu.hawaii.edu.intranet/files/upstf-student-success-bloom-1968.pdf.

5. If you had to pick just one manipulative/representation, which one would you pick?

I wouldn't. Don't get we wrong, I have representations that are my 'go to' models – in particular, bars and counters. These are usually the first representations I will try in order to support a pupil who is struggling to make the necessary connections or develop the required understanding. But when they don't work, I need others. Any teacher who reads this book and tries to focus on just one representation has missed a fundamental point: it is the flexibility of using multiple representations that compensate for the others' weaknesses that make them powerful. I want my pupils to understand what I understand about mathematics, but that doesn't mean I need them to think like me. I need to provide them with the models and structures that they require in order to reach this level of understanding, and so the more ways of thinking about the mathematics they are studying pupils have, the better.

6. My school/department doesn't have these manipulatives for use in the classroom. What can I do?

The United States Marine Corps has an official motto: 'Improvise, Adapt and Overcome'. In the absence of concrete manipulatives, adapt other things. I have used coloured pencils in place of bars and vectors. I have already mentioned teachers using sweets. Algebra tiles can be cut out of card (there are lots of templates available online). Failing all of that, there are virtual manipulatives which can be used either to model with pupils or for pupils to use if they have access to the appropriate technology. One of the best is the excellent www.mathsbot.com, created by Jonathan Hall (@ StudyMaths) – indeed, this is the source which provided the basis of most of the diagrams in this book. Given time, if your school/department can see that these approaches are working for pupils, they might be convinced to invest in some manipulatives for pupils to use.

7. Whenever I try to use these manipulatives, pupils just end up distracted by the manipulative and don't focus on the learning. How can I prevent this?

This is something I hear from teachers a lot. Unfortunately, there is no easy answer – as with any new approach in the classroom, pupils need time to get used to it and feel comfortable with it. There are many factors that will influence this, including the school culture and how this promotes pupils valuing their learning opportunities, the peer relationships and dynamics in the classroom, the prior understanding of pupils (both in terms of use of the representation and the concept) – the list goes on.

All I can say is that there are primary pupils up and down the country using these manipulatives to support the development of their own understanding, and therefore there is no reason why pupils of all ages shouldn't be able to achieve the same. The best advice I can give is to persevere – try to get to the point where the use of the manipulatives is a normal part of classroom practice, and eventually this should settle down.

8. My head of department/maths lead doesn't believe these approaches are worth bothering about, and insists I do things their way. What can I do?

This is always a tricky one to negotiate. Ultimately, I believe that the leader of an area should set the tone for how that area goes about its work, and the staff should take their cue from the head. But I also believe that leaders should listen to the viewpoints of their team and engage with new ideas, so if a leader isn't doing that, then it is legitimate to ask why. I would be tempted to talk with them, understand where their resistance comes from and see if I could get them to agree to allowing a trial of small changes in certain approaches. Then, if this works, I would take that back to them as a motivator for further change.

I am aware that this assumes a lot about how confident a teacher is in having this sort of conversation with their maths lead, which will be affected by their experience, how long they have worked in the department/school and so on. Ultimately, if it got to the point where I felt there was no way that I could use some classroom autonomy to introduce these approaches to support my pupils, if I felt the representations would benefit them, then I would have to consider moving on and finding an environment in which I would be more comfortable. But I appreciate this may not be an option for everyone.

9. How does this fit in with all the talk of 'mastery' that is being used in maths education?

We have to be careful when we talk about mastery in education, as it has come to mean certain things that it wasn't originally intended to mean. When it was first used in the early 20th century, mastery spoke of a curriculum model that could be applied to any subject, so the vast majority of learners could attain a high level of understanding in whatever was being taught. Mark McCourt provides an excellent outline of the history of the mastery curriculum.*

In addition to a mastery curriculum, we can talk about mathematics teaching for mastery approaches. This is a more recent development and is perhaps what most people mean when they talk about mastery in English schools. Different people have put forward different views, but they generally agree on the importance of the use of concrete and pictorial models to support progress in pupil understanding. Therefore, the approaches and concepts set out in this book can be considered fairly central to teaching for mastery in the mathematics classroom.

10. I am nervous about employing these approaches in the classroom. Is there somewhere I can go to see them being used in practice?

Absolutely. If you are in a secondary school, the first thing I would do is to contact some of your local primary schools – the use of representations seems to be much stronger in primaries than secondaries (in England anyway). There are also local Maths Hubs (www.mathshubs.org.uk), most of which will have teaching for mastery specialists at both primary and secondary level, who will have engaged with representation and structure and can share their practice with you. They may even have

* See Mark McCourt, Teaching for Mastery – Part 1, *Emaths* [blog] (2 October 2016). Available at: https://markmccourt.blogspot.co.uk/2016/10/teaching-for-mastery-part-1.html and Teaching for Mastery – Part 2, *Emaths* [blog] (21 October 2016). Available at: https://markmccourt.blogspot.co.uk/2016/10/teaching-for-mastery-part-2.html.

work groups around representation and structure or the use of manipulatives that you could get involved in, which would enable you to develop these approaches with the support of others in the work group. If not, the NCETM (www.ncetm.org.uk), which has oversight of the Maths Hub programme, has a large collection of materials – including videos and case studies – that may be helpful.

There are also a number of hands-on training opportunities in using manipulatives and representations. Some of the best are offered by La Salle Education, run by the great Mark McCourt. They include courses on mastery, bar modelling, and the concrete, pictorial, abstract and language approach. Details of these can be found on their website (http://completemaths.com).

I hope this chapter answers any questions you might have about the ideas and approaches discussed in the book. If they didn't, you can always contact the organisations I mention in question 10 above, or tweet them to me (@MrMattock) or to the maths community in general on Twitter – someone will have the answer. My sincere thanks for reading!

Glossary

Addend – A number that is added to another.

Argand diagram – A representation that plots complex numbers on a two-dimensional plane using one axis to represent the real part and the other axis to represent the irrational part.

Algorithm – A process or a set of instructions that are followed to produce a result.

Array – An arrangement of objects in two dimensions using columns and rows.

Base (of a number system) – The number of unique digits used to represent numbers in that system. The most well-known are binary (base 2), denary/decimal (base 10) and hexadecimal (base 16).

Cardinal numbers (cardinality) – Broadly speaking, numbers used to represent a count of how many objects make up a number or set – for example, one, two, three.

Chunking – An approach to division that involves removing large multiples of the **divisor** until zero is reached.

Continuous numbers – Numbers that can take all values and can be considered as connected to each other without a break.

Denominator – The value at the bottom of a fraction.

Discrete numbers – Numbers that can only take certain values and can be considered as separate objects to each other.

Dividend – In a division calculation, the number to be divided by another number (the **divisor**).

Divisor – In a division calculation, the number to be divided into another number (the **dividend**).

Equation – A mathematical statement showing that two things are equal. Often used specifically to describe problems where a variable is unknown and the values can be found using analytical or numerical approaches.

Exchange – The act of changing something for another, or others, of total equal value.

Expand – The process of applying the distributive law by multiplying a value over a sum or difference.

Factor – A positive whole number that divides into another number to produce a whole number answer.

Factorise – The process of writing a value or expression as the product of two or more **factors**.

Formula – A relationship between two or more variables.

Hypotenuse – The label given to the longest side in a right-angled triangle.

Identity element – The value that produces no effect when combined with any other under a given operation. The identity element of addition is 0, because 0 added to anything doesn't affect the thing. The identity element of multiplication is 1, because multiplying anything by 1 doesn't affect the thing.

Index (pl. indices) – An alternative word for the power to which a number is raised.

Integer – A whole number that can be positive, negative or zero.

Inverse (operation) – The operation that when applied to a value reverses the effect of a different operation applied to the same value. Subtraction is the inverse operation of addition because subtracting 4 (say) from a value reverses the effect of adding 4 to a value. Division is the inverse operation of multiplication because dividing a number by 4 reverses the effect of multiplying a number by 4.

Inverse (value) – A value that when combined with another under a given operation produces the **identity element** under that operation.

Irrational number – A number that cannot be expressed as the ratio of two **integers** (i.e. as a fraction).

Numerator – The value at the top of a fraction.

Ordered-pair – Two numbers grouped so that a change of order changes the meaning. Fractions and coordinates are both examples of ordered-pairs of **integers**.

Ordinal numbers (ordinality) – Numbers used to indicate position in a series (order) – for example, first, second, third.

Proportion – A relationship between quantities based on one quantity being a constant multiple of another.

Quotient – In a division calculation, the result of dividing a **dividend** by a **divisor**.

Rational number – A number that can be expressed as the fraction of two **integers** – for example, $\frac{1}{2}$, $\frac{2}{3}$ and 5.

Rounding – Reducing the accuracy of a given value due to physical requirements (i.e. rounding 42.2 pence as it is impossible to have 0.2 pence) or to reflect the accuracy of a measurement.

Standard index form – A way of writing numbers, usually very large or very small numbers, by indicating the size of the highest place in the number, with the size of smaller places given after a decimal point. Also called scientific notation.

Surd – An **irrational number**, usually used to mean the irrational root of a positive **integer** value.

Vector – A quantity that has a size (magnitude) and acts in a certain direction. Common examples include force and acceleration. The opposite of a vector is a scalar, which has size only (such as mass, area, etc.).

There's more to maths than finding the right answers

– what's much more important is understanding where they come from

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Geoff Wake, Professor of Mathematics Education and Convenor of the Centre for Research in Mathematics Education, University of Nottingham

Peter Mattock has been teaching maths for over a decade. He is a specialist leader of education (SLE) and an accredited secondary maths professional development lead, who regularly presents at conferences across the country. Peter also develops teaching for mastery in the secondary school classroom, having been part of the first cohort of specialists trained in mastery approaches by the National Centre for Excellence in the Teaching of Mathematics (NCETM). **@MrMattock**

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